Written Test for PhD Entrance IIT Palakkad
Jun 21, 2021

Read these instructions carefully before you begin:

- This written test comprises 6 sections. The first two sections on Real Analysis and Linear Algebra are mandatory for everyone. You have to attempt one or more sections from the next four sections depending on the specialization(s) you have opted for in the application form.
- Please justify every step in your answers. You can freely use standard results taught in your courses. For instance, you can say "as continuous functions on a compact set are bounded, it follows that....".
- This test works on an honour basis. You are not allowed to take help, consult with anyone or refer to the internet/textbooks during the course of this test. This is exactly like an exam but happening outside the exam hall and without supervision. Dishonesty in this written test will be grounds for disqualification.
- Please attempt the answers in A4 sheets of paper. Scan them using a scanning app like Microsoft Office Lens or similar. Please send only one pdf file. The naming convention is **ApplicationNO-NAME-Test.pdf**. For example, if your application number is PhD-MM-184-300 and your name is John Doe then name your file as *MM-184-300-John-Test.pdf*.

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1 Real Analysis

1. Construct a sequence $\{x_n\}$ in $[0,1]$ such that the set of all subsequential limits of $\{x_n\}$ is precisely $[0,1]$.
2. We say a sequence a_n is $o\left(\frac{1}{n}\right)$ if
$$\lim_{n\to\infty} na_n \to 0.$$
Show that if a_n is $o\left(\frac{1}{n}\right)$ then
$$\lim_{n\to\infty}\left(1+\frac{1}{n}+a_n\right)^n = e.$$
3. Let $f:(0,1)\to\mathbb{R}$ be a uniformly continuous function. Is f bounded? Prove or give an explicit counter-example.
4. Let $f:[-a,a]\to\mathbb{R}$ be a continuous function such that
$$\int_{-t}^{t} f(x)dx = 0$$
for each $t\in[0,a]$. Show that f is an odd function.

2 Linear Algebra

1. Let F be a subspace of the vector space $\mathbb{R}^d$. Suppose that F contains a vector all whose components are positive. Show that F is spanned by its probability vectors. (By a *probability vector* we mean, a vector all whose components are non-negative and add up to 1.)

2. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a map with the property that T maps linear subspaces into linear subspaces, then is it true that T is a linear map? Justify.

3. Let V be a finite dimensional vector space over $\mathbb{R}$ and let $T : V \to V$ be a linear transformation. Suppose $T \circ T$ is the identity map and -1 is *not* an eigenvalue of T. What can you say about T?

4. Let V be a finite dimensional real inner product space and let the associated norm be $\|\cdot\|$. Suppose we are given $u, v \in V$. Prove that u and v are orthogonal iff

$$\|u\| \leq \|u + av\| \quad \forall a \in \mathbb{R}.$$

3 Algebra

1. How many invertible 4×4 matrices are there with entries from $\mathbb{Z}_{17}$?

2. Let G be a group and H be a normal subgroup of G. Is it always true that G is isomorphic to $H \times \frac{G}{H}$?

3. Let G be a group of order 1055. Is there any normal subgroup of order 211 in G?

4. Let K be a field and $K^\times = K \backslash \{0\}$. Prove that every finite multiplicative subgroup of $K^\times$ is cyclic.

4 Complex Analysis

1. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function whose real part is a polynomial in x and y. Show that f is a polynomial in z.

2. Prove or disprove: there exists some open subset U of $\mathbb{C}$ and a sequence $c_0, c_1, c_2, c_3, \ldots$ of complex numbers such that:

$$\overline{z} = c_0 + c_1 z + c_2 z^2 + \ldots$$

   holds for $z \in U$ i.e., the power series on the right hand side converges and equals the complex conjugate function on U.

3. Write down (with justification), the radius of convergence of the power series, obtained by the Taylor expansion of the analytic functions about the stated point, in each of the following cases:

a) $f(z) = \frac{(z+20)(z+21)}{(z-20i)^{21}(z^2+z+1)}$ about $z = 0$, and

b) the principal branch of the logarithm about the point $-20 + 20i$.

4. Let $f : \mathbb{D} \to \mathbb{D}$ be an analytic function. Suppose for some $z_0 \in \mathbb{D}$, we have $f(z_0) = z_0$ and $f'(z_0) = 1$. What can you say about f?

5 Topology

1. Is there a continuous injective map from the circle S^1 to the real line $\mathbb{R}$?

2. Let A be any countable subset of $\mathbb{R}^2$. Prove that $\mathbb{R}^2 \setminus A$ is connected.

3. Recall that the Tietze extension theorem says: For any closed subspace A of a normal space X, any continuous function $f : A \to [a, b]$ has a continuous extension $g : X \to [a, b]$. Show that Tietze extension theorem implies the Urysohn's lemma.

4. Let (X, d) be a connected metric space with more than one point. Prove that the set X is uncountable.

6 Probability and Mathematical Statistics (Answer any four)

1. Suppose Bob thinks of an integer X from $\mathcal{X} := \{1, \ldots, M\}$ according to a probability distribution p_X. Alice tries to guess it by asking questions only of the form 'is $X = x$?', where x is any integer from $\mathcal{X}$ chosen uniformly random. Also suppose that her guesses are independent of her previous guesses.

   a) What is the expected number of guesses required for Alice to guess it correctly?

   b) Given that Alice guesses it correctly in her first guess itself, what is the probability that the number Bob thought of was 1?

2. Let $X_1, \ldots, X_n$ be i.i.d. random variables following a distribution with mean μ and variance σ^2.

   a) Use Chebyshev's inequality to find a bound for $\Pr(|\overline{X} - \mu| \leq \epsilon)$, where $\epsilon > 0$ and $\overline{X}$ is the sample mean of $X_1, \ldots, X_n$.

   b) Use the result in the previous part to prove the (weak) Law of Large Numbers.

   c) Suppose that we wish to get an idea of the number of people to be tested to get closer to the true proportion of people infected by COVID-19. Use the bound in first part to find a bound on the number of persons to be tested on a large population so that 99% percent certain that we are within 0.001 to the true proportion of infected people. (Hint: Variance of a Bernoulli random variable is at most 1/4)

3. Solve the third part of the second problem using interval estimation technique (Hint: CLT)

4. Show that, if X_n is a Poisson random variable with parameter n, then $X_n \overset{d}{\approx} N(n, n)$. (Hint: CLT)

5. Suppose that $U_1, \ldots, U_n$ are n i.i.d. random variables uniform on $(0, 1)$. Define

$$X = \text{number of } i : U_i \leq p,$$

where $0 < p < 1$. Find the distribution of X.

6. Let $X_1, \ldots, X_n$ be i.i.d. random variables following $\mathcal{N}(\mu, \sigma^2)$. Find the distribution of $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$? What can you say when the random variables are not normally distributed?